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LOG-CONVEX TRAPEZOIDAL APPROXIMATION OF AN ELEMENTARY INTEGRAL

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Log - convex trapezoidal approximation of an elementary integral

by

J. van de Lune & N.M. Temme

ABSTRACT

The integral $\int_0^1 x^s dx$, $s > 0$, is approximated by the canonical trapezoidal rule

$$T_n(s) = \frac{1}{2n} \{ \sum_{k=0}^{n-1} (k/n)^s + \sum_{k=1}^{n} (k/n)^s \}$$

and the log - convexity of $\{T_n(s)\}_{n=1}^{\infty}$ is studied, with s as a fixed parameter. The investigations are based on an integral representation of $T_n(s)$ and it is proved that the sequence $\{T_n(s)\}_{n=1}^{\infty}$ is log - convex (in n) for $1 < s < 3$ and $5 < s < 7$.

KEY WORDS & PHRASES: *Approximate quadrature, trapezoidal rule, convex sequences, Euler gamma function*

0. INTRODUCTION

We consider the *canonical trapezoidal approximations*

$$(0.1)\qquad T_n := T_n(s) := \frac{1}{2}\left(\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^s + \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s\right)$$

of the integral $\int_0^1 x^s dx$, where s is any *(fixed)* positive *real* number.

In [2] it was shown that for $s > 1$ (resp. $0<s<1$) the sequence $\{T_n\}_{n=1}^{\infty}$ is decreasing (resp. increasing), whereas somewhat later it was shown in [3] that for $s = 0(1)7$ and $s \geq 8$ this sequence even has the much stronger property of being *convex*.

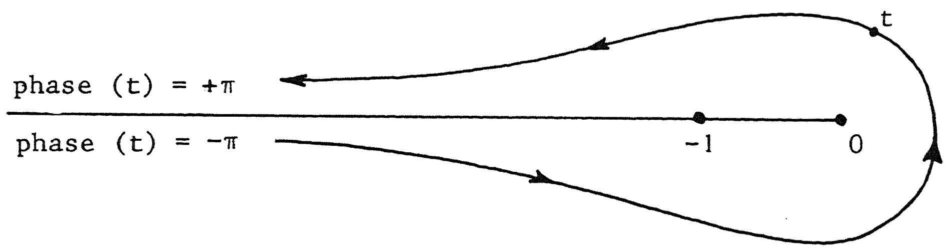
In [4; p. 8] the first named author conjectured that for all $s > 1$ the sequence $\{T_n\}_{n=1}^{\infty}$ is *logarithmically convex*, i.e. $T_n^2 \leq T_{n-1}T_{n+1}$ for all $n \geq 2$. The main goal of this note is to prove the correctness of this conjecture for the intervals $1 < s < 3$ and $5 < s < 7$.

1. PRELIMINARIES

Our starting point is Hankel's integral representation of the reciprocal of Euler's gamma function (cf. WHITTAKER & WATSON [6; pp. 244-245] or SANSONE & GERRETSEN [5; pp. 201-204])

$$(1.1)\qquad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\oint e^t t^{-s} dt, \qquad s \in \mathbb{C}$$

where $\oint$ denotes integration along a contour as depicted below:



For any $p > 0$ we substitute $t = pw$ in (1.1), replace s by $s + 1$ and obtain

$$(1.2)\qquad p^{s} = \frac{\Gamma(s+1)}{2\pi i} \oint e^{pw} w^{-s-1} dw, \qquad s \in \mathbb{C}.$$

Setting $p = \frac{k}{n}$, $k = 1(1)n$, we obtain by summation over k

$$(1.3)\qquad T_n = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^{w}-1}{w} \frac{w}{2n} \frac{e^{\frac{w}{n}}+1}{e^{\frac{w}{n}}-1} w^{-s-1} dw, \qquad s > 0.$$

Letting $n \to \infty$ it follows that

$$\frac{1}{s+1} = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^{w}-1}{w} w^{-s-1} dw, \qquad s > 0$$

(a result obtainable in various other ways; compare Section 4) so that (1.3) may be rewritten as

$$(1.4)\qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^{w}-1}{w} H(\tfrac{w}{n}) w^{-s-1} dw, \qquad s > 0$$

where

$$H(z) = \frac{z}{2}\frac{e^{z}+1}{e^{z}-1} - 1 = z\left(\frac{1}{e^{z}-1} - \frac{1}{z} + \frac{1}{2}\right).$$

It is well known that (cf. SANSONE & GERRETSEN [5; p. 88])

$$(1.5)\qquad \frac{1}{e^{z}-1} - \frac{1}{z} + \frac{1}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < 2\pi$$

from which it is clear that the *(even)* function $H(z)$ has a zero of order 2 at $z = 0$. With this in mind we rewrite (1.4) as follows

$$(1.6)\qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^{w}-1}{w} \left(\frac{1}{w^2} H(\tfrac{w}{n})\right) w^{1-s} dw, \qquad s > 0.$$

2. THE CASE $1 < s < 2$.

For $1 < s < 2$ (so that $-1<1-s<0$) we may, by the regularity of $w^{-2} H(\frac{w}{n})$ at $w = 0$, contract the contour of integration in (1.6) to the negative real axis so that by a standard argument, using the fact that $H(z)$ is an *even* function,

$$(2.1)\qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi}\int_0^\infty \frac{1-e^{-x}}{x}\, H(\tfrac{x}{n})\, x^{-s-1}dx, \qquad 1 < s < 2.$$

Substituting $x = nu$ and writing $\frac{1-e^{-nu}}{nu} = \int_0^1 e^{-nuv}dv$ we may write (2.1) as

$$(2.2)\qquad T_n - \frac{1}{s+1} = \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi}\int_0^\infty \left(\int_0^1 e^{-nuv}dv\right) H(u)\, u^{-s-1}du.$$

Since $\sin(s-1)\pi > 0$ for $1 < s < 2$ and $H(u) > 0$ for $u > 0$, we find, by the general theory of log-convex functions (cf. ARTIN [1]), that the sequence $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ is log-convex, a result which is even stronger than the previously announced assertion that $\{T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (1,2)$.

Similarly one may show that $\{\frac{1}{s+1} - T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (0,1)$.

3. INTERMEZZO: A SPECIAL PROPERTY OF $H(u) = u(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2})$

In the previous section we transformed (2.1) into (2.2) and then concluded that $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ is log-convex for all $s \in (1,2)$. In this section we will show that this result may also be obtained directly from (2.1) by observing that the function $H(\frac{1}{x})$, $x > 0$, has the remarkable property of being log-convex on $\mathbb{R}^+$. As a matter of fact we will prove the following

<u>THEOREM 3.1.</u> *There exists a constant* $\alpha_0 > 2.863$ *such that for every (fixed)* $\alpha \in (0,\alpha_0]$ *the function* $\phi_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, *defined by* $\phi_\alpha(x) := H(x^{-\alpha})$, $x > 0$, *is log-convex on* $\mathbb{R}^+$.

<u>PROOF.</u> In order to prove the log-convexity of ϕ_α on $\mathbb{R}^+$ we proceed by brute force, at the same time inviting the reader to invent a nicer proof.

Writing

$$\psi(x) := \log \phi_\alpha(x) = \log H(x^{-\alpha})$$

we have

$$\psi''(x) = \alpha\, u^2 + \frac{AB-C^2}{A^2}$$

where

$$u := \frac{1}{x}\ ,$$

$$v := u^\alpha,$$

$$A := (e^v-1)^{-1} - \frac{1}{v} + \frac{1}{2}\ ,$$

$$B := 2\,\alpha^2u^2v^2e^{2v}(e^v-1)^{-3} - \alpha^2u^2v^2e^v(e^v-1)^{-2}$$

$$- \alpha(\alpha+1)u^2ve^v(e^v-1)^{-2} - \alpha(\alpha-1)\,\frac{u^2}{v}\ ,$$

$$C := \alpha\, uve^v(e^v-1)^{-2} - \alpha\,\frac{u}{v}\ .$$

It clearly suffices to show that $\psi''(x) > 0$ for all $x \in \mathbb{R}^+$ so that (since $\alpha > 0$) we may just as well prove that

$$\frac{\psi''(x)}{\alpha u^2} = 1 + \frac{A_1B_1-\alpha C_1^2}{A_1^2} > 0$$

where (u and v being defined as above)

$$A_1 := A \text{ (as defined above)},$$

$$B_1 := 2\,\alpha\, v^2e^{2v}(e^v-1)^{-3} - \alpha\, v^2e^v(e^v-1)^{-2}$$

$$- (\alpha+1)ve^v(e^v-1)^{-2} - \frac{\alpha-1}{v}\ ,$$

$$C_1 := ve^v(e^v-1)^{-2} - \frac{1}{v}\ .$$

Hence, it suffices to show that for all $x \in \mathbb{R}^+$

$$A_1^2 + A_1B_1 > \alpha\, C_1^2 .$$

Multiplying both sides of this inequality by $v^2(e^v-1)^4$ we arrive at the equivalent inequality

$$v^2(e^v-1)^2 + (v^2-2v)\ (e^v-1)^3 + (1-\frac{v}{2})^2\ (e^v-1)^4 +$$

$$+ (v+(\frac{v}{2}-1)(e^v-1))(2\alpha v^3e^{2v}- \alpha v^3e^v(e^v-1) - (\alpha+1)v^2e^v(e^v-1)$$

$$- (\alpha-1)(e^v-1)^3) > \alpha(v^4e^{2v}-2v^2e^v(e^v-1)^2 + (e^v-1)^4).$$

This inequality may be written in the equivalent form

$$\sum_{k=0}^{4} P_k(v)e^{kv} > 0 \tag{3.1}$$

where

$$P_0(v) = \frac{\alpha+1}{2} + \frac{v}{4} ,$$

$$P_1(v) = -\ (\alpha+1) + (3\alpha+1)v + \frac{3\alpha+1}{2}\ v^2 + \frac{\alpha}{2}\ v^3,$$

$$P_2(v) = -\ \frac{5+12\alpha}{2}\ v ,$$

$$P_3(v) = \alpha + 1 + (3\alpha+1)v - \frac{3\alpha+1}{2}\ v^2 + \frac{\alpha}{2}\ v^3,$$

$$P_4(v) = -\ \frac{\alpha+1}{2} + \frac{v}{4} .$$

Now we write the left hand side of (3.1) in the form $\Sigma_{n=0}^{\infty}\ c_n v^n$ and observe that $c_0 = c_1 = 0$ for all α. For $n \geq 2$ one may verify that

$$n!c_n = (\alpha+1)(-1+3^n-2^{2n-1}) +$$

$$+ n((3\alpha+1) - (12\alpha+5)2^{n-2} + (3\alpha+1)3^{n-1} + 4^{n-2}) +$$

$$+ (3\alpha+1)\frac{n(n-1)}{2}(1-3^{n-2}) + \frac{\alpha}{2} n(n-1)(n-2)(1+3^{n-3}) =$$

$$=: \alpha\, a(n) + b(n) ,$$

where

$$a(n) := -1 + 3^n - 2^{2n-1} + 3n - 3n2^n + n3^n + \frac{n(n-1)}{2}(3-3^{n-1}) +$$

$$+ \frac{n(n-1)(n-2)}{2}(1+3^{n-3}),$$

$$b(n) := -1 + 3^n - 2^{2n-1} + n - 5n2^{n-2} + n3^{n-1} + n4^{n-2} +$$

$$+ \frac{n(n-1)}{2}(1-3^{n-2}).$$

It is a matter of routine to show that

$$a(n) = 0 \quad \text{for} \quad n \le 8,$$

$$a(n) < 0 \quad \text{for} \quad n \ge 9,$$

$$b(n) = 0 \quad \text{for} \quad n \le 6,$$

$$b(n) > 0 \quad \text{for} \quad n \ge 7,$$

and

$$\min_{n\ge 9} -\frac{b(n)}{a(n)} = -\frac{b(24)}{a(24)} = 2.863\,921\ \ldots,$$

from which it follows that for $0 < \alpha < 2.8639$ we have $c_n = 0$ for $n \le 6$ and $c_n > 0$ for $n \ge 7$, which proves the theorem.

REMARK. It is not known to us which α_0^* is the largest number such that $H(x^{-\alpha})$ is log-convex on $\mathbb{R}^+$ for all $\alpha \in (0,\alpha_0^*]$. Numerical computations show that $H(x^{-3})$ is *not* log-convex on all of $\mathbb{R}^+$ so that $(2.863<)\ \alpha_0^* < 3$.

4. FURTHER PREPARATIONS

In order to carry our analysis somewhat further we need some auxiliary formulas. In (1.2) let $p \downarrow 0$ (keeping s fixed and > 0) and it follows that

$$\oint w^{-s-1}dw = 0, \qquad s > 0. \tag{4.1}$$

Another way of proving this formula is as follows. In (1.3) put $n = 1$ so that (for $s>0$)

$$\begin{aligned} T_1(s) = \frac{1}{2} &= \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w+1}{2} w^{-s-1}dw = \\ &= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint e^w w^{-s-1}dw + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint w^{-s-1}dw = \\ &= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\,\frac{2\pi i}{(s+1)} + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint w^{-s-1}dw, \end{aligned} \tag{4.2}$$

and it follows again that $\oint w^{-s-1}dw = 0$ for $s > 0$.
Our next important auxiliary result is

LEMMA 4.1. *For any positive integer* N *we have*

$$H(z) = z\left(\frac{1}{e^z-1} - \frac{1}{z} + \frac{1}{2}\right) = P_N(z) + (-1)^N z^{2N+2}\,\mu_N(z) \tag{4.3}$$

where

$$P_N(z) := \sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} z^{2k} \tag{4.4}$$

and

$$\mu_N(z) := \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2m^2)(2\pi m)^{2N}}\,. \tag{4.5}$$

PROOF. In order to prove this lemma we apply Taylor's formula as described in WHITTAKER & WATSON [6; p. 93]. We observe that (compare (1.5))

$$(4.6)\qquad P_N(z) = \sum_{k=1}^{N} \frac{H^{(2k)}(0)}{(2k)!} z^{2k} \quad\text{and}\quad H^{(2k+1)}(0) = 0,$$

so that

$$(4.7)\qquad (-1)^N \mu_N(z) = \frac{1}{2\pi i} \oint \frac{H(w)}{(w-z)w^{2N+2}}\, dw,$$

where $\oint$ denotes counter clockwise integration along a closed contour containing the points $w = 0$ and $w = z$ in its interior and such that it does not encircle any of the points $w = k.2\pi i$, $k \in \mathbb{Z} \setminus \{0\}$. A standard application of the calculus of residues then yields

$$(-1)^N \mu_N(z) = \frac{1}{2\pi i} \oint \frac{\frac{1}{e^w-1} - \frac{1}{w} + \frac{1}{2}}{(w-z)w^{2N+1}}\, dw =$$

$$(4.8)\qquad = -\sum_{m=1}^{\infty} \left\{ \frac{1}{(2\pi i m - z)(2\pi i m)^{2N+1}} + \frac{1}{(-2\pi i m - z)(-2\pi i m)^{2N+1}} \right\} =$$

$$= (-1)^N \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2 m^2)(2\pi m)^{2N}},$$

and the lemma follows.

REMARKS.

1) We note that Lemma 4.1 also holds true for $N = 0$. In this case we have the well-known formula

$$H(z) = \sum_{m=1}^{\infty} \frac{2z^2}{z^2+4\pi^2 m^2}.$$

2) As an immediate consequence of Lemma 4.1 we have for any fixed $N > 0$

$$\mu_N(x) = O(x^{-2}), \qquad x \to \infty.$$

3) $\mu_N(z)$ is regular at $z = 0$.

LEMMA 4.2. *For any fixed* $N > 0$ *the function* $\mu_N(\frac{1}{x})x^{-2N-2}$ *is log-convex on* $\mathbb{R}^+$.

PROOF. In order to see this we write

$$\mu_N(\tfrac{1}{x})x^{-2N-2} = 2\sum_{m=1}^{\infty} \frac{x^{-2N-2}}{(x^{-2}+4\pi^2m^2)(2\pi m)^{2N}} =$$

$$= 2\sum_{m=1}^{\infty} \frac{1}{(1+4\pi^2m^2x^2)(2\pi mx)^{2N}}$$

and observe that every term of this series is log-convex on $\mathbb{R}^+$. Indeed, for any (fixed) $a > \frac{1}{8}$ the function

$$\phi_a(x) = -\log(1+x^2) - 2a\log x$$

is convex on $\mathbb{R}^+$.

5. THE CASE $2 < s < 3$

From

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} H(\tfrac{w}{n}) w^{-s-1} dw$$

we obtain by means of the results of the previous section (for $2<s<3$)

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (H(\tfrac{w}{n})-P_1(\tfrac{w}{n}))w^{-s-1}dw +$$

$$+ \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} P_1(\tfrac{w}{n})w^{-s-1}dw .$$

Since $P_1(z) = \frac{z^2}{12}$ we thus find that

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \left(\frac{w}{n}\right)^4 (-1)^1 \mu_1\left(\frac{w}{n}\right) w^{-s-1} dw +$$

$$+ \frac{\Gamma(s+1)}{2\pi i} \frac{1}{12n^2} \oint (e^w-1) w^{-s} dw =$$

$$= \frac{1}{s+1} + \frac{s}{12n^2} - \frac{\Gamma(s+1)}{\pi} \sin(s-3)\pi \int_0^\infty \frac{1-e^{-t}}{t} \left(\frac{t}{n}\right)^4 \mu_1\left(\frac{t}{n}\right) t^{-s-1} dt.$$

In Section 4 it was shown that $x^{-4}\mu_1(x^{-1})$ is log-convex on $\mathbb{R}^+$ so that for any $t > 0$, $\left(\frac{t}{n}\right)^4 \mu_1\left(\frac{t}{n}\right)$ is log-convex as a function of $n \in \mathbb{N}$. Since $\sin(s-3)\pi < 0$ for $2 < s < 3$ it follows that $\{T_n - \frac{1}{s+1} - \frac{s}{12n^2}\}_{n=1}^\infty$ is log-convex (in n) for any fixed $s \in (2,3)$, a result which is even stronger than the previously announced log-convexity of $\{T_n\}_{n=1}$.

6. SOME REMARKS ON THE GENERAL CASE: $2N < s < 2(N+1)$

Similarly as before we have

$$T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} P_N\left(\frac{w}{n}\right) w^{-s-1} dw +$$

$$+ (-1)^N \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \left(\frac{w}{n}\right)^{2N+2} \mu_N\left(\frac{w}{n}\right) w^{-s-1} dw =$$

$$= \frac{1}{s+1} + I_1(n) + I_2(n), \text{ say.}$$

According to the preliminaries in Section 4 we have

$$I_1(n) = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} \left(\sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} \left(\frac{w}{n}\right)^{2k} \right) w^{-s-1} dw =$$

$$= \sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} \frac{\Gamma(s+1)}{\Gamma(s-2k+2)} \frac{1}{n^{2k}},$$

and, similarly as before,

$$I_2(n) = (-1)^N \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} \left(\frac{w}{n}\right)^{2N+2} \mu_N\left(\frac{w}{n}\right) w^{-s-1} dw =$$

$$= (-1)^N \frac{\Gamma(s+1)\sin(s-2N-1)\pi}{\pi} \int_0^\infty \frac{1-e^{-t}}{t} \left(\frac{t}{n}\right)^{2N+2} \mu_N\left(\frac{t}{n}\right) t^{-s-1}\, dt,$$

the last integral being convergent at $t = 0$ since $(2N+2) - s - 1 > -1$ and at $t = \infty$ since $-1 + (2N+2) - 2 - s - 1 < -1$. We now observe that

$$\begin{aligned}
&N \text{ even and } 2N + 1 < s < 2N + 2 &&\Rightarrow\ (-1)^N \sin(s-2N-1)\pi > 0,\\
&N \text{ even and } 2N < s < 2N + 1 &&\Rightarrow\ \qquad '' \qquad\qquad\quad < 0,\\
&N \text{ odd and } 2N < s < 2N + 1 &&\Rightarrow\ \qquad '' \qquad\qquad\quad > 0,\\
&N \text{ odd and } 2N + 1 < s < 2N + 2 &&\Rightarrow\ \qquad '' \qquad\qquad\quad < 0.
\end{aligned}$$

Hence, whenever we can show that $\{I_2(n)\}_{n=1}^\infty$ is log-convex then $\{T_n\}_{n=1}^\infty$ is log-convex if $(-1)^N \sin(s-2N-1)\pi > 0$. It follows that our approach can only be successful if $2N + 1 < s < 2N + 3$, where N is *even*.

7. THE CASE $5 < s < 7$

We first assume $5 < s < 6$ so that

$$T_n(s) - \frac{1}{s+1} = \frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \text{log-convex (in } n).$$

Hence, in order to show the log-convexity of $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ it suffices to show the log-convexity of $\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4}\}_{n=1}^\infty$. Since $5 < s < 6$ it is easily seen that this in its turn is a consequence of the log-convexity of $\{\frac{1}{n^2} - \frac{1}{3n^4}\}_{n=1}^\infty$, the verification of which is a matter of routine. Now let $6 < s < 7$, so that by the results of Section 6 it suffices to show the log-convexity of

$$\left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)(s-2)(s-3)(s-4)}{42\,720n^6}\right\}_{n=1}^{\infty},$$

which, using the assumption $6 < s < 7$, is an easy consequence of the log-convexity of $\{\frac{1}{n^2} - \frac{7}{12n^4} + \frac{3}{89n^6}\}_{n=1}^{\infty}$, a (though tedious) matter of routine.

REMARK. For $9 < s < 10$ we would have to verify the log-convexity of

$$\left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)\ldots(s-4)}{42\,720n^6} - \frac{s(s-1)\ldots(s-6)}{1\,209\,600\,n^8}\right\}_{n=1}^{\infty}$$

whereas for still larger values of s it seems practically unfeasible (if true) to prove the log-convexity (in n) of forms of such a complexity.

REFERENCES

[1] ARTIN, E., *Einführung in die Theorie der Gammafunktion*, Teubner, Leipzig, 1931.

[2] LUNE, J. VAN DE, *Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers*, Report ZW 39/75, Mathematical Centre, Amsterdam.

[3] LUNE, J. VAN DE & M. VOORHOEVE, *Convex approximation of integrals*, Report ZW 85/77, Mathematical Centre, Amsterdam.

[4] LUNE, J. VAN DE & M. VOORHOEVE, *Some problems on log-convex approximation of certain integrals*, Report ZN 85/78, Mathematical Centre, Amsterdam.

[5] SANSONE, G. & J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, Noordhoff, Groningen, 1960.

[6] WHITTAKER, E.T. & G.N. WATSON, *A course of Modern Analysis*, Cambridge, 1952.